Cohomology of the complement to an elliptic arrangement

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Abstract. We consider the complement to an arrangement of hyperplanes in a cartesian power of an elliptic curve and describe its cohomology with coefficients in a nontrivial rank one local system.

1 Introduction

We start with a cartesian power E^k of an elliptic curve E and a nontrivial rank one local system on E^k. We consider an arrangement of elliptic hyperplanes in E^k and describe the cohomology of its complement with coefficients in the local system.We show that the cohomology is nontrivial only in degree k. We present each cohomology class by a unique closed holomorphic differential form. Our forms are elliptic analogs of the Arnold-Brieskorn-Orlik-Solomon logarithmic differential forms representing cohomology classes of the complement to an arrangement of hyperplanes in an affine space. For the elliptic discriminantal arrangement our forms are the forms considered in [2–4] to solve the KZB equations in hypergeometric integrals and to construct Bethe eigenfunctions to the elliptic Calogero-Moser operators.

To simplify the exposition, we first consider in Sections 2 and 3 the case of an elliptic discriminantal arrangement, then in Sections 4 and 5 we consider arbitrary elliptic arrangements.

2   Cohomology of an elliptic discriminantal arrangement

Fix a natural number k and $\tau \in \mathbb{C}$, $\operatorname{Im} \tau > 0$. Denote $\Lambda = \tau \mathbb{Z} + \mathbb{Z} \subset \mathbb{C}$. The group $\Gamma = \mathbb{Z} \oplus \mathbb{Z}$ acts on $\mathbb{C}$ by transformations $(l, m) : t \mapsto t + l\tau + m$. The action on each factor gives an action of Γ^k on $\mathbb{C}^k$. Denote by $p : \mathbb{C}^k \to \mathbb{C}^k / \Gamma^k$ the canonical projection onto the space of orbits. We have $\mathbb{C}^k / \Gamma^k = E^k$, where E is the elliptic curve $\mathbb{C} / \Gamma$.

For each representation ρ of Γ^k on a vector space W we get a vector bundle over E^k with a flat connection, which is $(\mathbb{C}^k \times W) / \Gamma^k \to \mathbb{C}^k / \Gamma^k$. In particular, we may fix complex numbers $\boldsymbol{w} = (w_1, \ldots, w_k)$, take $W = \mathbb{C}$, and $\rho_{\boldsymbol{w}}(\gamma) = e^{2\pi \sqrt{-1}(w_1 l_1 + \cdots + w_k l_k)}$ for $\gamma = (l_1, m_1) \times \cdots \times (l_k, m_k)$. This line bundle over E^k with the flat connection will be denoted by $\mathcal{L}_{\boldsymbol{w}}$.

We say that the numbers $\boldsymbol{w} = (w_1, \ldots, w_k)$ are *discriminantal convenient* if for any subset $I \subset \{1, \ldots, k\}$ the sum $\sum_{i \in I} w_i$ is not in Λ.

Fix distinct complex numbers $z = (z_1, \ldots, z_n)$. The *discriminantal* arrangement $\mathcal{C}_z$ in $\mathbb{C}^k$ with parameters z is the arrangement of hyperplanes:

$$
\begin{aligned}
H_i^a &: t_i - z_a = 0, & i &= 1, \ldots, k, \ a = 1, \ldots, n; \\
H_{ij} &: t_i - t_j = 0, & 1 &\leqslant i < j \leqslant k.
\end{aligned}
$$

Let M_z denote its complement $\mathbb{C}^k - \cup_{H \in \mathcal{C}_z} H$.

The Γ^k-orbit of $\mathcal{C}_z$ is the infinite arrangement $\mathcal{C}_{z, \Gamma^k} = \{\gamma(H) \mid \gamma \in \Gamma^k, H \in \mathcal{C}_z\}$. Denote by M_{z, Γ^k} its complement $\mathbb{C}^k - \cup_{\gamma \in \Gamma^k, H \in \mathcal{C}_z} \gamma(H)$. Denote by $\tilde{M}_{z, \tau} \subset E^k$ the image of M_{z, Γ^k} under the projection p.

Theorem 2.1. *Assume that the numbers $\boldsymbol{w}$ are discriminantal convenient and $z_1, \ldots, z_n$ project to distinct points of E. Then $H^\ell(\tilde{M}_{z, \tau}; \mathcal{L}_{\boldsymbol{w}}) = 0$ for $\ell \neq k$ and $H^k(\tilde{M}_{z, \tau}; \mathcal{L}_{\boldsymbol{w}})$ is canonically isomorphic to $H^k(M_z; \mathbb{C})$, the k-th cohomology group with trivial coefficients of the complement in $\mathbb{C}^k$ to the discriminantal arrangement.*

Here $H^*(\tilde{M}_{z, \tau}; \mathcal{L}_{\boldsymbol{w}})$ denotes the cohomology of $\tilde{M}_{z, \tau}$ with coefficients in the local system of horizontal sections of $\mathcal{L}_{\boldsymbol{w}}$. Theorem 2.1 is proved in Section 5.

The space $\tilde{M}_{z, \tau}$ is a $K(\pi, 1)$-space and our theorem describes the cohomology of the fundamental group of $\tilde{M}_{z, \tau}$ with coefficients in $\mathcal{L}_{\boldsymbol{w}}$. Notice that the fundamental group of $\tilde{M}_{z, \tau}$ is a subgroup of the pure elliptic braid group with $n + k$ strings.

The cohomology $H^k(M_z; \mathbb{C})$ of the complement to the discriminantal arrangement in $\mathbb{C}^k$ are presented be explicit logarithmic forms by the Arnold-Brieskorn-Orlik-Solomon theory. Below we describe logarithmic differential forms representing elements of $H^k(\tilde{M}_{z, \tau}; \mathcal{L}_{\boldsymbol{w}})$. Those forms

were used in [2–4] to give integral hypergeometric representations for solutions of the KZB equations with values in a tensor product of highest weight representations of a simple Lie algebra and to construct Bethe eigenfunctions of elliptic Calogero-Moser operators.

3 Differential forms of a discriminantal arrangement

In this section we follow [4]. Theorem 3.3 is new.

3.1 Combinatorial space

An *ordered k-forest* is a graph with no cycles, with k edges, and a numbering of its edges by the numbers $1, 2, \ldots, k$. We consider the ordered k-forests on the vertex set of symbols $\{z_1, \ldots, z_n, t_1, \ldots, t_k\}$. An ordered k-forest T is *admissible* if all $t_1, \ldots, t_k$ are among the vertices of T and each connected component of T has exactly one vertex from the subset $\{z_1, \ldots, z_n\}$.

Let $\mathcal{A}_n^k$ be the complex vector space generated by the admissible ordered k-forests, modulo the following relations:

R1 $T_1 = -T_2$ if T_1 and T_2 have the same underlying graph, and the order of their edges differ by a transposition;

R2

$$ \quad (a, b \in \{1, \ldots, k\}), $$

that is, the sum of three k-forests that locally (i.e. their subgraphs spanned by 3 vertices) differ as above, but are otherwise identical, is 0.

A linear map ϕ of $\mathcal{A}_n^k$ to a vector space W is called a *representation* of $\mathcal{A}_n^k$. Suppose we are given a vector space W and a vector $\phi(T) \in W$ for every admissible k-forest T. This data induces a representation if the assignment $T \mapsto \phi(T)$ respects relations R1 and R2.

3.2 Rational representation

Let e be an edge of an admissible forest T. The connected component of T, containing e, has exactly one vertex, say z_a, from the set $\{z_1, \ldots, z_n\}$. Denote by $h(e)$ and $t(e)$ the head and tail of the edge e, i.e. the vertices adjacent to e, farther resp. closer to the vertex z_a.

Fix distinct complex numbers $z = (z_1, \ldots, z_n)$. To an admissible forest T with ordered edges $e_1, \ldots, e_k$, we assign a closed holomorphic differential k-form $\phi_{rat}(T)$ on M_z by the formula

$$\phi_{rat}(T) = \wedge_{i=1}^{k} d \log(h(e_i) - t(e_i)).$$

This assignment defines a representation of $\mathcal{A}_n^k$ on the space of k-forms on M_z, see [1, 5]. By [1, 5], the representation is an isomorphism onto its image, see Proposition 2.1 in [4]. We denoted the image by $\mathcal{A}_z^k$. The assignment to a form of its cohomology class gives a linear map $\mathcal{A}_n^k \to H^k(M_z; \mathbb{C})$.

Theorem 3.1 ([1,5]). *The map $\mathcal{A}_n^k \to H^k(M_z; \mathbb{C})$ is an isomorphism.*

3.3 Theta representation.

For $z, \tau \in \mathbb{C}$, Im $\tau > 0$, the first Jacobi theta function is defined by the infinite product

$$\theta(z) = \theta(z, \tau) = \sqrt{-1} e^{\pi\sqrt{-1}(\tau/4-z)}(x; q) \left(\frac{q}{x}; q\right)(q; q),$$

$$q = e^{2\pi\sqrt{-1}\tau}, \quad x = e^{2\pi\sqrt{-1}z}, \qquad (y; q) = \prod_{j=0}^{\infty}(1 - yq^j),$$

[9]. It is an entire holomorphic function of z satisfying

$$\theta(z + 1, \tau) = -\theta(z, \tau), \quad \theta(z + \tau, \tau) = -e^{-\pi\sqrt{-1}\tau - 2\pi\sqrt{-1}z}\theta(z, \tau),$$
$$\theta(-z, \tau) = -\theta(z, \tau).$$

By $\theta'(z, \tau)$ we will mean the derivative in the z variable. Define

$$\sigma_w(t) = \sigma_w(t, \tau) = \frac{\theta(w - t, \tau)}{\theta(w, \tau)\theta(t, \tau)} \cdot \theta'(0, \tau).$$

The listed properties of the theta function yield that the function σ – viewed as a function of t – has simple poles at the points of $\Lambda \subset \mathbb{C}$, as well as the properties

$$\sigma_w(t + 1, \tau) = \sigma_w(t, \tau), \qquad \sigma_w(t + \tau, \tau) = e^{2\pi\sqrt{-1}w}\sigma(t, \tau),$$
$$\mathrm{Res}_{t=0}\, \sigma_w(t, \tau) = 1.$$

We also have

$$\sigma_{w_1+w_2}(t-u)\sigma_{w_2}(s-t) - \sigma_{w_2}(s-u)\sigma_{w_1}(t-u) + \sigma_{w_1}(t-s)\sigma_{w_1+w_2}(s-u) = 0,$$

see for example, [4].

Fix discriminantal convenient complex numbers $\boldsymbol{w} = (w_1, \ldots, w_k)$ and distinct complex numbers $z = (z_1, \ldots, z_n)$. For $i = 1, \ldots, k$, we say that t_i has *weight* w_i.

Let T be an admissible forest and v a vertex of T. The connected component of T, containing v, has exactly one vertex, say z_a, from the set $\{z_1, \ldots, z_n\}$. We define the *branch* $B(v)$ of v to be the collection of those vertices u for which the unique path connecting u with z_a contains v. By definition $v \in B(v)$. The *load* $L(v)$ of a vertex v in the forest T is defined to be the sum of the weights of the vertices in $B(v)$.

To an admissible forest T with ordered edges $e_1, \ldots, e_k$, we assign a closed holomorphic differential k-form $\phi_\theta(T)$ on M_{z,Γ^k} by the formula

$$\phi_\theta(T) = \wedge_{i=1}^{k} \sigma_{L(h(e_i))}(h(e_i) - t(e_i), \tau) \, d(h(e_i) - t(e_i)).$$

Notice that if $\boldsymbol{w}$ are discriminantal convenient, then the load of each vertex $h(e_i)$ does not lie in Λ and the form is well-defined.

Theorem 3.2 ([4]). *Assume that $\boldsymbol{w}$ is discriminantal convenient and $z_1, \ldots, z_n$ project to distinct points of E. Then the assignment $T \mapsto \phi_\theta(T)$ defines a representation of $\mathcal{A}_n^k$ on the space of k-forms on M_{z,Γ^k}. The representation is an isomorphism onto its image, denoted by $\mathcal{A}_{z,\Gamma^k}^k$. Each element of $\mathcal{A}_{z,\Gamma^k}^k$ descends to a closed holomorphic differential form on $\tilde{M}_{z,\tau}$ with values in $\mathcal{L}_{\boldsymbol{w}}$.*

The assignment to a form of its cohomology class defines a linear map $\mathcal{A}_n^k \to H^k(\tilde{M}_{z,\tau}; \mathcal{L}_{\boldsymbol{w}})$. In Section 5 the following theorem will be proved.

Theorem 3.3. *Assume that the numbers $\boldsymbol{w}$ are discriminantal convenient and $z_1, \ldots, z_n$ project to distinct points of E. Then the map $\mathcal{A}_n^k \to H^k(\tilde{M}_{z,\tau}; \mathcal{L}_{\boldsymbol{w}})$ is an isomorphism.*

Theorems 3.1 and 3.3 imply the second statement of Theorem 2.1.

According to [7], $\dim \mathcal{A}_n^k = \sum_{m_1 + \cdots + m_n = k} m_1! \ldots m_n!$.

4   Transversal elliptic hyperplanes

4.1   Elliptic hyperplanes in E^k

Denote $\mathcal{E} = E^k$. Any $k \times k$-matrix $C \in GL(k, \mathbb{Z})$ defines an isomorphism $\mathcal{E} \to \mathcal{E}$, $(\tilde{t}_1, \ldots, \tilde{t}_k) \mapsto (\sum_j \tilde{t}_j c_{j1}, \ldots, \sum_j \tilde{t}_j c_{jk})$. The collection $\sum_j \tilde{t}_j c_{j1}, \ldots, \sum_j \tilde{t}_j c_{jk}$ will be called *coordinates* on $\mathcal{E}$.

Let $\tilde{t}_1', \ldots, \tilde{t}_k'$ be coordinates on $\mathcal{E}$. Fibers of the projection $\mathcal{E} \to E^\ell$ along the last $k - \ell$ coordinates will be called *elliptic $k - \ell$-planes* in $\mathcal{E}$,

in particular, elliptic $k - 1$-planes are *elliptic hyperplanes*. The fibers of the same projection will be called *parallel $k - \ell$-planes*.

An elliptic $k - \ell$-plane is defined by equations $\tilde{t}'_i = \tilde{z}_i$, $i = 1, \ldots, \ell$, for suitable $\tilde{z}_i \in E$. Each elliptic $k - \ell$-plane is isomorphic to $E^{k-\ell}$ as an algebraic variety.

Lemma 4.1. *The normal bundle of an elliptic $k - \ell$-plane in $\mathcal{E}$ is trivial.*

4.2 Intersection of $\ell \leqslant k$ transversal elliptic hyperplanes

We say that $\ell \leqslant k$ elliptic hyperplanes $\tilde{H}_1, \ldots, \tilde{H}_\ell$ intersect transversally, if they are defined by equations

$$\tilde{t}_1 a_{1j} + \cdots + \tilde{t}_k a_{kj} - \tilde{z}_j = 0, \qquad j = 1, \ldots, \ell, \qquad (4.1)$$

and the rank of the $\ell \times k$-matrix $a = (a_{ij})$ equals ℓ.

By standard theorems, see for example [8], there are coordinates $\tilde{t}'_1, \ldots, \tilde{t}'_k$ on $\mathcal{E}$ such that system (4.1) is equivalent to a system

$$d_j \tilde{t}'_j - \tilde{z}'_j = 0, \qquad j = 1, \ldots, \ell, \qquad (4.2)$$

where $\tilde{z}'_j \in E$, $d_j \in \mathbb{Z}_{>0}$ and $d_j | d_{j+1}$ for $j = 1, \ldots, \ell - 1$. Therefore, the intersection X of ℓ transversal hyperplanes in $\mathcal{E}$ consists of $(d_1 \ldots d_\ell)^2$ parallel elliptic $k - \ell$-planes.

Let $\boldsymbol{w} = (w_1, \ldots, w_k)$ be complex numbers and $\mathcal{L}_{\boldsymbol{w}}$ the line bundle over $\mathcal{E}$ with a flat connection defined in Section 2. We say that $\boldsymbol{w}$ are *convenient* for $\mathcal{E}$, if there are no nonzero $\mathcal{L}_{\boldsymbol{w}}$-valued holomorphic differential k-forms on $\mathcal{E}$.

Lemma 4.2. *Complex numbers $\boldsymbol{w}$ are convenient for $\mathcal{E}$ if and only if $\boldsymbol{w} \notin \Lambda^k$.*

Proof. It is enough to prove the lemma for $k = 1$. If $k = 1$ and ω is an $\mathcal{L}_{\boldsymbol{w}}$-valued holomorphic 1-form, then it is 1-periodic and has the Fourier series expansion. The expansion easily implies the required statement. $\qquad\square$

Similarly, we say that $\boldsymbol{w}$ are *convenient* for the transversal intersection X with $\dim X = k - \ell > 0$, if there are no nonzero $\mathcal{L}_{\boldsymbol{w}}$-valued holomorphic differential $k - \ell$-forms on any of the parallel $k - \ell$-planes composing X. Let X be defined by equations (4.1).

Lemma 4.3. *Complex numbers $\boldsymbol{w}$ are convenient for the transversal intersection X, $\dim X > 0$, if and only if there exist integers $l_1, \ldots, l_k$ such that $\sum_{i=1}^k l_i a_{ij} = 0$ for $j = 1, \ldots, \ell$, and $l_1 w_1 + \cdots + l_k w_k \notin \Lambda$.*

Proof. The proof of Lemma 4.3 is the same as the proof of Lemma 4.2. □

Lemma 4.4. *If the numbers $\boldsymbol{w}$ are convenient for $\mathcal{E}$, then $H^*(\mathcal{E}, \mathcal{L}_{\boldsymbol{w}}) = 0$. If $\boldsymbol{w}$ are convenient for the transversal intersection X, $\dim X > 0$, then $H^*(X, \mathcal{L}_{\boldsymbol{w}}|_X) = 0$.*

Proof. The lemma follows from the Kunneth formula and the fact that the cohomology of a circle with coefficients in a nontrivial local system is zero. □

4.3   Differential forms of k transversal hyperplanes in $\mathcal{E}$

This section contains the main construction of the paper.

Let k transversal elliptic hyperplanes $\tilde{H}_1, \ldots, \tilde{H}_k$ in $\mathcal{E}$ be given by equations

$$\sum_{i=1}^{k} \tilde{t}_i a_{ij} = \tilde{z}_j \qquad j = 1, \ldots, k, \tag{4.3}$$

where $\tilde{t}_1, \ldots, \tilde{t}_k$ are coordinates on $\mathcal{E}$, $a = (a_{ij})$ is an integer matrix (with nonzero determinant) and $\tilde{z}_1, \ldots, \tilde{z}_k$ are some points of E.

For a complex number c, we denote by $\tilde{c}$ its projection to E. In particular, $\tilde{0} \in E$ is the projection of 0. For given complex numbers $\boldsymbol{w} = (w_1, \ldots, w_k)$, we consider the system of equations

$$\sum_{j=1}^{k} a_{ij} \tilde{v}_j = \tilde{w}_i \qquad i = 1, \ldots, k, \tag{4.4}$$

with respect to the unknown $\tilde{v}_1, \ldots, \tilde{v}_k \in E$. We say that $\boldsymbol{w}$ is *admissible* for $\tilde{H}_1, \ldots, \tilde{H}_k$ if any coordinate $\tilde{v}_j$ of any solution of (4.4) is not equal to $\tilde{0}$.

Lemma 4.5. *Assume that the numbers $\boldsymbol{w}$ are convenient for each of the transversal intersections X_j, $j = 1, \ldots, k$, where X_j is the intersection of the elliptic hyperplanes $\tilde{H}_1, \ldots, \tilde{H}_{j-1}, \tilde{H}_{j+1}, \ldots, \tilde{H}_k$, then $\boldsymbol{w}$ are admissible for $\tilde{H}_1, \ldots, \tilde{H}_k$.*

Proof. Let $\tilde{v}_1, \ldots, \tilde{v}_k$ be any solution of system (4.4). Let $l_1, \ldots, l_k$ be integers such that $\sum_{i=1}^{k} l_i a_{ij} = 0$ for $j = 2, \ldots, k$, and $l_1 w_1 + \cdots + l_k w_k \notin \Lambda$. Then $\tilde{0} \neq \sum_i l_i \tilde{w}_i = \sum_{ij} l_i a_{ij} \tilde{v}_j = \sum_i l_i a_{i1} \tilde{v}_1$. Hence $\tilde{v}_1 \neq \tilde{0}$. Similarly we prove that $\tilde{v}_2, \ldots, \tilde{v}_k$ are not equal to $\tilde{0}$. □

Let $\boldsymbol{w}$ be admissible for $\tilde{H}_1, \ldots, \tilde{H}_k$. Fix complex numbers $z_1, \ldots, z_k$ whose projections to E are $\tilde{z}_1, \ldots, \tilde{z}_k$. For any integers A_i, B_i, C_i, D_i with $i = 1, \ldots, k$, we consider two systems of equations:

$$\sum_{i=1}^{k} u_i a_{ij} = A_j \tau + B_j + z_j, \qquad j = 1, \ldots, k, \tag{4.5}$$

and

$$\sum_{j=1}^{k} a_{ij} v_j = C_i \tau + D_i + w_i, \qquad i = 1, \ldots, k. \tag{4.6}$$

The first system is with respect to complex numbers $\boldsymbol{u} = (u_1, \ldots, u_k)$ and the second system is with respect to complex numbers $\boldsymbol{v} = (v_1, \ldots, v_k)$.

To the solution $\boldsymbol{v} = (v_1, \ldots, v_k)$ of (4.6), we assign the meromorphic k-form on $\mathbb{C}^k$,

$$\omega_{\boldsymbol{v}}(t, \tau) = \omega_{\boldsymbol{v}}(t_1, \ldots, t_k, \tau) \tag{4.7}$$
$$= \det a \; e^{-2\pi\sqrt{-1}\sum_{i=1}^{k} C_i t_i} \prod_{j=1}^{k} \sigma_{v_j} \left(\sum_{i=1}^{k} t_i a_{ij} - z_j, \tau \right) dt_1 \wedge \cdots \wedge dt_k.$$

The form is well-defined since the numbers $\boldsymbol{w}$ are admissible for $\tilde{H}_1, \ldots, \tilde{H}_k$.

Lemma 4.6. *The form $\omega_{\boldsymbol{v}}(t, \tau)$ descends to an $\mathcal{L}_{\boldsymbol{w}}$-valued meromorphic form on $\mathcal{E}$, i.e. $\omega_{\boldsymbol{v}}(t + \gamma, \tau) = \rho_{\boldsymbol{w}}(\gamma)\omega_{\boldsymbol{v}}(t, \tau)$ for $\gamma \in \Gamma^k$.*

Lemma 4.7. *The form $\omega_{\boldsymbol{v}}(t, \tau)$ does not change if $\boldsymbol{v}$ is changed by an element of Λ^k.*

Lemma 4.8. *Let $\boldsymbol{u} = (u_1, \ldots, u_k)$ be the solution of system (4.5). Then*

$$\omega_{\boldsymbol{v}}(t_1 + u_1, \ldots, t_k + u_k, \tau)$$
$$= M(\boldsymbol{u}, \boldsymbol{v}) \det a \; e^{-2\pi\sqrt{-1}\sum_{i=1}^{k} C_i t_i} \prod_{j=1}^{k} \sigma_{v_j} \left(\sum_{i=1}^{k} t_i a_{ij}, \tau \right) dt_1 \wedge \cdots \wedge dt_k,$$

where

$$M(\boldsymbol{u}, \boldsymbol{v}) = e^{2\pi\sqrt{-1}\sum_{i=1}^{k}(A_i v_i - C_i u_i)}.$$

For a complex number c, we shall write $c = c_{\mathbb{R}} + \tau c_{\tau}$ with $c_{\mathbb{R}}, c_{\tau} \in \mathbb{R}$.

Lemma 4.9. *We have*

$$\sum_{i=1}^{k}(A_i v_i - C_i u_i) = \sum_{i=1}^{k}(A_i v_{i,\mathbb{R}} - B_i v_{i,\tau}) + \sum_{i=1}^{k}(u_i w_{i,\tau} - z_i v_{i,\tau}).$$

Proof. $\sum_i A_i v_i = \sum_i (A_i v_{i,\mathbb{R}} + \tau A_i v_{i,\tau})$ and $\sum_i C_i u_i = \sum_i (\sum_j a_{ij} v_{j,\tau} - w_{i,\tau}) u_i = \sum_{ij} u_{i,\mathbb{R}} a_{ij} v_{j,\tau} + \tau \sum_{ij} u_{i,\tau} a_{ij} v_{j,\tau} - \sum_i w_{i,\tau} u_i = \sum_j (B_j + z_{j,\mathbb{R}}) v_{j,\tau} + \tau \sum_j (A_j + z_{j,\tau}) v_{j,\tau} - \sum_i w_{i,\tau} u_i$. These equalities give the lemma. $\qquad\square$

If $\boldsymbol{u}$ is a solution of (4.5), then $p(\boldsymbol{u})$ is a solution (4.3). All solutions of (4.3) have this form. Similar relations hold for systems (4.6) and (4.4).

Each of the systems (4.3) and (4.4) has $(\det a)^2$ solutions. For each solution $\tilde{\boldsymbol{u}}$ of (4.3) we fix a solution $\boldsymbol{u}$ of (4.5) such that $p(\boldsymbol{u}) = \tilde{\boldsymbol{u}}$. We denote by $\mathcal{U}$ the constructed set of $(\det a)^2$ points $\boldsymbol{u} \in \mathbb{C}^k$. For each solution $\tilde{\boldsymbol{v}}$ of (4.4) we fix a solution $\boldsymbol{v}$ of (4.6) such that $p(\boldsymbol{v}) = \tilde{\boldsymbol{v}}$. We denote by $\mathcal{V}$ the constructed set of $(\det a)^2$ points $\boldsymbol{v} \in \mathbb{C}^k$.

Theorem 4.10. *The matrix $M = (M(\boldsymbol{u}, \boldsymbol{v}))_{\boldsymbol{u}\in\mathcal{U},\boldsymbol{v}\in\mathcal{V}}$ is nondegenerate.*

Proof. Let $M_1(\boldsymbol{u}, \boldsymbol{v}) = e^{2\pi\sqrt{-1}\sum_{i=1}^{k}(A_i v_{i,\mathbb{R}} - B_i v_{i,\tau})}$.
The matrix $M_1 = (M_1(\boldsymbol{u}, \boldsymbol{v}))_{\boldsymbol{u}\in\mathcal{U},\boldsymbol{v}\in\mathcal{V}}$ is obtained from M by multiplication by nondegenerate diagonal matrices. Thus, it is enough to prove that M_1 is nondegenerate.

The nondegeneracy of M_1 follows from the nondegeneracy of M_1 for $\boldsymbol{w} = 0$ and $\boldsymbol{z} = 0$, since the matrix M_1 for $\boldsymbol{w}, \boldsymbol{z}$ not necessarily equal to zero is obtained from the matrix M_1 with $\boldsymbol{w} = 0, \boldsymbol{z} = 0$ by multiplication by nondegenerate diagonal matrices.

By elementary row and column transformations, the pair of systems (4.5) and (4.6) can be reduced to the case of a diagonal matrix a. For a diagonal a and $\boldsymbol{w} = 0, \boldsymbol{z} = 0$ the nondegeneracy of M_1 is obvious. $\qquad\square$

Theorem 4.11. *Let $\boldsymbol{w}$ be admissible for $\tilde{H}_1, \ldots, \tilde{H}_k$. Let $\mathcal{U}$ be a set as above. Then there exist the unique differential k-forms $\omega_{\boldsymbol{u},\tilde{H}_1,\ldots,\tilde{H}_k}(\boldsymbol{t}, \tau)$, $\boldsymbol{u} \in \mathcal{U}$, such that each $\omega_{\boldsymbol{u},\tilde{H}_1,\ldots,\tilde{H}_k}(\boldsymbol{t}, \tau)$ is a $\mathbb{C}$-linear combination of forms $\omega_{\boldsymbol{v}}(\boldsymbol{t}, \tau)$, $\boldsymbol{v} \in \mathcal{V}$, and for any $\boldsymbol{u}, \boldsymbol{u}' \in \mathcal{U}$ we have the followings expansion,*

$$\omega_{\boldsymbol{u},\tilde{H}_1,\ldots,\tilde{H}_k}(\boldsymbol{t} + \boldsymbol{u}', \tau)$$

$$= (\delta_{\boldsymbol{u},\boldsymbol{u}'} + \mathcal{O}(\boldsymbol{t}))\, d\log\left(\sum_{i=1}^{k} t_i a_{i1}\right) \wedge \cdots \wedge d\log\left(\sum_{i=1}^{k} t_i a_{ik}\right),$$

where $\mathcal{O}(\boldsymbol{t})$ is a function holomorphic at $\boldsymbol{t} = 0$ and $\mathcal{O}(\boldsymbol{0}) = 0$.

Proof. The theorem is a direct corollary of Theorem 4.10.   □

Given transversal $\tilde{H}_1, \ldots, \tilde{H}_k$, the set $\mathcal{U}$ is not unique, each point $\boldsymbol{u}' \in \mathcal{U}$ can be shifted by any element $\gamma = (l_1\tau + m_1, \ldots, l_k\tau + m_k)$ of Λ^k.

Lemma 4.12. *Assume that exactly one point $\boldsymbol{u}'$ of the set $\mathcal{U}$ is replaced with a point $\boldsymbol{u}'' = \boldsymbol{u}' + \gamma$. Consider the set of differential forms assigned to the new set $\mathcal{U}$ by Theorem 4.11. Then $\omega_{\boldsymbol{u}'', \tilde{H}_1, \ldots, \tilde{H}_k}(t, \tau) = e^{2\pi\sqrt{-1}(w_1 l_1 + \ldots + w_k l_k)} \omega_{\boldsymbol{u}', \tilde{H}_1, \ldots, \tilde{H}_k}(t, \tau)$ and all other differential forms $\omega_{\boldsymbol{u}, \tilde{H}_1, \ldots, \tilde{H}_k}(t, \tau)$, $\boldsymbol{u} \in \mathcal{U}$, remain unchanged.*

4.3.1 The residue of $\omega_{\boldsymbol{u}, \tilde{H}_1, \ldots, \tilde{H}_k}$ Let H be a hyperplane in $\mathbb{C}^k$ defined by an equation $t_1 a_{1j} + \cdots + t_k a_{kj} = A_j\tau + B_j + z_j$, where $j \in \{1, \ldots, k\}$ and A_j, B_j are some integers, *cf.* (4.5) and (4.7). We have $p(H) = \tilde{H}_j$. Let $\boldsymbol{u}$ be a point of $\mathcal{U}$ and $\omega_{\boldsymbol{u}, \tilde{H}_1, \ldots, \tilde{H}_k}$ the corresponding differential form. We denote by $\eta_{\boldsymbol{u}}$ the residue of $\omega_{\boldsymbol{u}, \tilde{H}_1, \ldots, \tilde{H}_k}$ at H.

Lemma 4.13. *Assume that a vector $\gamma = (l_1\tau + m_1, \ldots, l_k\tau + m_k) \in \Lambda^k$ is tangent to H, i.e. $\sum_i (l_i\tau + m_i)a_{ij} = 0$. Then for all $t \in H$, we have $\eta_{\boldsymbol{u}}(t + \gamma) = e^{2\pi\sqrt{-1}(w_1 l_1 + \cdots + w_k l_k)}\eta_{\boldsymbol{u}}(t)$. That is, the form $\eta_{\boldsymbol{u}}$ defines an $\mathcal{L}_{\boldsymbol{w}}$-valued differential form over the elliptic hyperplane $p(H) = \tilde{H}_j \subset \mathcal{E}$.*

Now we choose H in Lemma 4.13 so that $\boldsymbol{u} \in H$.

For $i \neq j$, the intersection $\tilde{H}_i \cap \tilde{H}_j$ is a collection of parallel elliptic $k - 2$-planes. We denote by $\tilde{H}_i^{(j)}$ that elliptic $k - 2$-plane which contains $\tilde{\boldsymbol{u}} = p(\boldsymbol{u})$. Then $\tilde{H}_1^{(j)}, \ldots, \tilde{H}_{j-1}^{(j)}, \tilde{H}_{j+1}^{(j)}, \ldots, \tilde{H}_k^{(j)}$ are transversal elliptic hyperplanes in $\tilde{H}_j$.

Theorem 4.14. *We have $\eta_{\boldsymbol{u}}(t) = (-1)^{j-1}\omega_{\boldsymbol{u}, \tilde{H}_1^{(j)}, \ldots, \tilde{H}_{j-1}^{(j)}, \tilde{H}_{j+1}^{(j)}, \ldots, \tilde{H}_k^{(j)}}.$*

Proof. The difference of the right hand side and the left hand side defines an $\mathcal{L}_{\boldsymbol{w}}$-valued form on $\tilde{H}_j$ with logarithmic singularities along $\tilde{H}_1 \cap \tilde{H}_j, \ldots, \tilde{H}_{j-1} \cap \tilde{H}_j, \tilde{H}_{j+1} \cap \tilde{H}_j, \ldots, \tilde{H}_k \cap \tilde{H}_j$. The difference has zero $k - 1$-iterated residues at all points. Therefore, the difference vanishes due to the following lemma.   □

Lemma 4.15. *Assume that for every $i = 1, \ldots, k$, we have a finite set of parallel hyperplanes $\{\tilde{H}_i^{l_i} \mid l_i \in L_i\}$ in $\mathcal{E}$. Assume that the hyperplanes $\tilde{H}_1^{l_1}, \ldots, \tilde{H}_k^{l_k}$ intersect transversally. Assume that numbers $\boldsymbol{w}$ are convenient for the transversal intersection of $\tilde{H}_1^{l_1}, \ldots, \tilde{H}_k^{l_k}$.*

Let Ω be an $\mathcal{L}_{\boldsymbol{w}}$-valued meromorphic differential k-form on $\mathcal{E}$ with logarithmic singularities at the union of all hyperplanes $\{\tilde{H}_i^{l_i} | i = 1, \ldots, k, l_i \in L_i\}$. Assume that Ω has zero k-iterated residues at all points of $\mathcal{E}$. Then Ω is the zero form.

Proof. The proof is by induction on k. If $k = 1$, then Ω is regular on E. Since $\boldsymbol{w}$ are convenient, Ω vanishes.

Step of the induction. The residue of Ω at any hyperplane $\tilde{H}_j^{l_j}$ has the same properties as Ω: the residue has logarithmic singularities at the union of all intersections $\tilde{H}_i^{l_i} \cap \tilde{H}_j^{l_j}$, the residue has zero $k - 2$-iterated residue at any point. By the induction assumption, the residue of Ω at $\tilde{H}_j^{l_j}$ vanishes, hence, Ω is regular on $\mathcal{E}$ and Ω is the zero form due to the convenience of $\boldsymbol{w}$. $\square$

4.3.2 Example   Here is an example illustrating Theorem 4.11 for $k = 1$. Consider an analog of the pair of systems (4.3) and (4.4): $2\tilde{t} = 0$ and $2\tilde{v} = \tilde{w}$, where $w \notin \Lambda$. We can choose $\mathcal{U} = \{0, 1/2, \tau/2, 1/2 + \tau/2\}$ and $\mathcal{V} = \{w/2, w/2+1/2, w/2+\tau/2, w/2+1/2+\tau/2\}$. The differential forms ω_v, $\boldsymbol{v} \in \mathcal{V}$, given by formula (4.7), are

$$\omega_1 = 2\sigma_{w/2}(2t, \tau)dt, \quad \omega_2 = 2\sigma_{w/2+1/2}(2t, \tau)dt,$$
$$\omega_3 = 2e^{-2\pi\sqrt{-1}t}\sigma_{w/2+\tau/2}(2t,\tau)dt, \quad \omega_4 = 2e^{-2\pi\sqrt{-1}t}\sigma_{w/2+1/2+\tau/2}(2t,\tau)dt$$

Denote $\gamma = e^{-\pi\sqrt{-1}w}$. Then the differential forms ω_u, $\boldsymbol{u} \in \mathcal{U}$, given by Theorem 4.11, are

$$\tilde{\omega}_1 = \frac{1}{4}(\omega_1 + \omega_2 + \gamma\omega_3 + \gamma\omega_4), \quad \tilde{\omega}_2 = \frac{1}{4}(\omega_1 + \omega_2 - \gamma\omega_3 - \gamma\omega_4),$$

$$\tilde{\omega}_3 = \frac{1}{4}(\omega_1 - \omega_2 + \gamma\omega_3 - \gamma\omega_4), \quad \tilde{\omega}_4 = \frac{1}{4}(\omega_1 - \omega_2 - \gamma\omega_3 + \gamma\omega_4).$$

The forms $\tilde{\omega}_i$, $i = 1, \dots, 4$, define meromorphic $\mathcal{L}_{\boldsymbol{w}}$-valued differential forms on E. The form $\tilde{\omega}_1$ is regular on $\mathbb{C} - \Lambda$, has simple poles at Λ, has residue 1 at $t = 0$. The forms $\tilde{\omega}_2, \tilde{\omega}_3, \tilde{\omega}_4$ have similar properties with respect to the sets $\mathbb{C} - (1/2 + \Lambda), \mathbb{C} - (\tau/2 + \Lambda), \mathbb{C} - (\tau/2 + 1/2 + \Lambda)$ and points $1/2, \tau/2, \tau/2 + 1/2$, respectively. These properties imply that

$$\tilde{\omega}_1 = \sigma_w(t, \tau)dt, \qquad \tilde{\omega}_2 = \sigma_w(t - 1/2, \tau)dt,$$
$$\tilde{\omega}_3 = \sigma_w(t - \tau/2, \tau)dt, \qquad \tilde{\omega}_4 = \sigma_w(t - \tau/2 - 1/2, \tau)dt.$$

5   Arbitrary elliptic arrangement

5.1   An elliptic arrangement

An elliptic arrangement in $\mathcal{E} = E^k$ is a finite collection $\mathcal{C} = \{\tilde{H}_j\}_{j \in J}$ of elliptic hyperplanes. We fix coordinates $\tilde{t}_1, \dots, \tilde{t}_k$ on $\mathcal{E}$ and for every

$j \in J$ we fix an equation $\tilde{t}_1 a_{1j} + \cdots + \tilde{t}_k a_{kj} - \tilde{z}_j = 0$ defining the hyperplane $\tilde{H}_j$.

We denote by

$$\tilde{M}_{\mathcal{C}} = \mathcal{E} - \cup_{j \in J} \tilde{H}_j \, ,$$

the complement of the arrangement.

Consider the intersection of any $\ell \leqslant k$ transversal hyperplanes of $\mathcal{C}$. The intersection consists of a finite set of parallel elliptic $k - \ell$ planes. Each of these $k - \ell$-planes will be called an edge of $\mathcal{E}$. In particular, if $\ell = k$, then the 0-planes will be called *vertices* of $\mathcal{E}$.

For an edge X we denote $J_X = \{j \in\subset J \mid X \subset \tilde{H}_j\}$.

We denote by $\tilde{\mathcal{U}}$ the set of all vertices of $\mathcal{C}$. For every vertex $\tilde{\boldsymbol{u}} \in \tilde{\mathcal{U}}$ we choose a point $\boldsymbol{u} \in \mathbb{C}^k$ such that $p(\boldsymbol{u}) = \tilde{\boldsymbol{u}}$. The set of all chosen points in $\mathbb{C}^k$ is denoted by $\mathcal{U}$.

We say that complex numbers $\boldsymbol{w} = (w_1, \ldots, w_k)$ are *convenient* for the elliptic arrangement $\mathcal{C}$, if $\boldsymbol{w}$ are convenient for the intersection of every $\ell < k$ transversal hyperplanes $\tilde{H}_{j_1}, \ldots, \tilde{H}_{j_\ell}$ of $\mathcal{C}$ (in the sense of Section 4.2).

5.2   Differential k-forms of an elliptic arrangement

For a vertex $\tilde{\boldsymbol{u}} \in \tilde{\mathcal{U}}$, we denote by $\mathcal{C}_{\tilde{\boldsymbol{u}}} = \{\tilde{H}_j\}_{j \in I_{\tilde{\boldsymbol{u}}}}$ the subarrangement of all hyperplanes of $\mathcal{C}$ containing $\tilde{\boldsymbol{u}}$. In a small neighborhood of $\tilde{\boldsymbol{u}}$ the arrangement $\mathcal{C}_{\tilde{\boldsymbol{u}}}$ is isomorphic to a central arrangement of affine hyperplanes. We denote by $\mathcal{A}_{\tilde{\boldsymbol{u}}}^k$ the k-th graded component of the Orlik-Solomon algebra of that arrangement. More precisely, let $\mathcal{A}_{\tilde{\boldsymbol{u}}}^k$ be the complex vector space generated by symbols $(\tilde{H}_{j_1}, ..., \tilde{H}_{j_k})$ with $j_i \in J_{\tilde{\boldsymbol{u}}}$, subject to the relations:

(i) $(\tilde{H}_{j_1}, ..., \tilde{H}_{j_k}) = 0$ if $\tilde{H}_{j_1}, ..., \tilde{H}_{j_k}$ are not transversal;

(ii) $(\tilde{H}_{j_{\sigma(1)}}, ..., \tilde{H}_{j_{\sigma(k)}}) = (-1)^{|\sigma|} (\tilde{H}_{j_1}, ..., \tilde{H}_{j_k})$ for any $\sigma \in S_k$;

(iii) $\sum_{i=1}^{k+1} (-1)^i (\tilde{H}_{j_1}, ..., \widehat{\tilde{H}_{j_i}}, ..., \tilde{H}_{j_{k+1}}) = 0$ for any $k + 1$ elliptic hyperplanes of $\mathcal{C}_{\tilde{\boldsymbol{u}}}$.

We set

$$\mathcal{A}_{\mathcal{C}}^k = \oplus_{\tilde{\boldsymbol{u}} \in \tilde{\mathcal{U}}} \mathcal{A}_{\tilde{\boldsymbol{u}}}^k.$$

Let us fix $\boldsymbol{w} = (w_1, \ldots, w_k)$ convenient for $\mathcal{E}$. Let $\tilde{\boldsymbol{u}} \in \tilde{\mathcal{U}}$ and $\boldsymbol{u} \in \mathcal{U}$ be such that $p(\boldsymbol{u}) = \tilde{\boldsymbol{u}}$. Let $\tilde{H}_{j_1}, \ldots, \tilde{H}_{j_k}$ be any k transversal hyperplanes in $\mathcal{C}_{\tilde{\boldsymbol{u}}}$. Denote by $\omega_{\boldsymbol{u}, \tilde{H}_{j_l}, ..., \tilde{H}_{j_k}}(\boldsymbol{t}, \tau)$ the differential meromorphic k-form on $\mathbb{C}^k$ assigned by Theorem 4.11 to these k transversal hyperplanes and

denoted by $\omega_{\boldsymbol{u}, \tilde{H}_1, \ldots, \tilde{H}_k}(\boldsymbol{t}, \tau)$ in Theorem 4.11. We denote by $A_{\tilde{\boldsymbol{u}}}^k$ the complex vector space generated by the forms $\omega_{\boldsymbol{u}; \tilde{H}_{j_l}, \ldots, \tilde{H}_{j_k}}(\boldsymbol{t}, \tau)$. Notice that by Lemma 4.12, the space $A_{\tilde{\boldsymbol{u}}}^k$ does not depend on the choice of $\boldsymbol{u}$ such $p(\boldsymbol{u}) = \tilde{\boldsymbol{u}}$.

We denote by $A_{\mathcal{C}}^k$ the sum of vector spaces $A_{\tilde{\boldsymbol{u}}}^k$, $\tilde{\boldsymbol{u}} \in \tilde{\mathcal{U}}$.

Theorem 5.1.

(i) *The map* $\mathcal{A}_{\tilde{\boldsymbol{u}}}^k \to A_{\tilde{\boldsymbol{u}}}^k$, $(\tilde{H}_{j_l}, \ldots, \tilde{H}_{j_k}) \mapsto \omega_{\boldsymbol{u}; \tilde{H}_{j_l}, \ldots, \tilde{H}_{j_k}}(\boldsymbol{t}, \tau)$, *is an isomorphism of vector spaces.*

(ii) *We have* $A_{\mathcal{C}}^k = \oplus_{\tilde{\boldsymbol{u}} \in \tilde{\mathcal{U}}} A_{\tilde{\boldsymbol{u}}}^k$.

Proof. It is enough to prove that for any $k + 1$ elliptic hyperplanes of $\mathcal{C}_{\tilde{\boldsymbol{u}}}$, we have the elliptic Orlik-Solomon relation

$$\sum_{i=1}^{k+1} (-1)^i \omega_{\boldsymbol{u}, \tilde{H}_{j_1}, \ldots, \widehat{\tilde{H}_{j_i}}, \ldots, \tilde{H}_{j_{k+1}}} = 0. \tag{5.1}$$

The proof is by induction on k. If $k = 1$, the difference $\omega_{\boldsymbol{u}, \tilde{H}_1} - \omega_{\boldsymbol{u}, \tilde{H}_2}$ is regular on E and is the zero 1-form due to the convenience of $\boldsymbol{w}$.

Step of the induction. For every $i = 1, \ldots, k + 1$, the residue at $\tilde{H}_{j_i}$ of the left hand side in (5.1) is the left hand side of an elliptic Orlik-Solomon relation for an arrangement in $\tilde{H}_{j_i}$, see Theorem 4.14. By the induction assumption, the residue of the left hand side at $\tilde{H}_{j_i}$ is the zero $k - 1$-form. Hence, the left hand side in (5.1) is regular on $\mathcal{E}$ and vanishes due to the convenience of $\boldsymbol{w}$. $\square$

5.3   Cohomology of the complement

Every form $\omega \in A_{\mathcal{C}}^k$ induces a holomorphic $\mathcal{L}_{\boldsymbol{w}}$-valued k-form $p_*(\omega)$ on the complement $\tilde{M}_{\mathcal{C}}$ of the elliptic arrangement $\mathcal{C}$. The image of $A_{\mathcal{C}}^k$ will be denoted by $p_*(A_{\mathcal{C}}^k)$. The assignment to $p_*(\omega)$ its cohomology class $[p_*(\omega)]$ defines a linear map $\iota : A_{\mathcal{C}}^k \to H^k(\tilde{M}_{\mathcal{C}}; \mathcal{L}_{\boldsymbol{w}})$. Here $H^*(\tilde{M}_{\mathcal{C}}; \mathcal{L}_{\boldsymbol{w}})$ denotes the cohomology of $\tilde{M}_{\mathcal{C}}$ with coefficients in the local system of horizontal sections of $\mathcal{L}_{\boldsymbol{w}}$.

Theorem 5.2. *Assume that* $\boldsymbol{w}$ *are convenient for* $\mathcal{C}$. *Then* $H^\ell(\tilde{M}_{\mathcal{C}}; \mathcal{L}_{\boldsymbol{w}}) = 0$ *for* $\ell \neq k$ *and* $\iota_{\mathcal{C}} : A_{\mathcal{C}}^k \to H^k(\tilde{M}_{\mathcal{C}}; \mathcal{L}_{\boldsymbol{w}})$ *is an isomorphism.*

Proof. We need the following lemmas. $\square$

Lemma 5.3. *The map* $\iota_{\mathcal{C}} : A_{\mathcal{C}}^k \to H^k(\tilde{M}_{\mathcal{C}}; \mathcal{L}_{\boldsymbol{w}})$ *is a monomorphism.*

Proof. For a central affine arrangement of hyperplanes in $\mathbb{C}^k$, the k-th homology group of the complement with trivial coefficients is generated by k-dimensional tori located near the vertex of the arrangement and corresponding to the k-flags of the arrangement, see Section 4.4 in [7]. The nondegenerate pairing between the top degree cohomology of the complement with trivial coefficients and the top degree homology is given by the integrals of the Orlik-Solomon differential forms over the tori. The integrals are nothing else but the multiple residues of the differential forms at the flags of the arrangement. Locally at $\tilde{u} \in \tilde{\mathcal{U}}$, the arrangement $\mathcal{C}_{\tilde{u}}$ is isomorphic to a central affine arrangement. The k-dimensional tori of that central arrangement, considered as k-dimensional tori in a small neighborhood of $\tilde{u}$ in $\tilde{M}$ induces a vector subspace $H_{\tilde{u},k} \subset H_k(\tilde{M}; \mathcal{L}_{\boldsymbol{w}})$. The pairing between $H_{\tilde{u},k}$ and $A_{\tilde{u}'}$ is zero if $\tilde{u} \neq \tilde{u}'$ and the pairing is nondegenerate if $\tilde{u} = \tilde{u}'$. $\square$

Lemma 5.4. *Theorem 5.2 is true for $k = 1$.*

Proof. The lemma follows from the convenience of $\boldsymbol{w}$ and the exact sequence for the pair $\tilde{M}_{\mathcal{C}} \subset E$. $\square$

Let j_0 be an element of J. We consider the following three elliptic arrangements: $\mathcal{C}, \mathcal{C}', \mathcal{C}''$, where $\mathcal{C}' = \{\tilde{H}_j\}_{j \in J - \{j_0\}}$ and $\mathcal{C}''$ is the elliptic arrangement induced by $\mathcal{C}$ on $\tilde{H}_{j_0}$.

Lemma 5.5. *We have an exact sequence*

$$0 \to A_{\mathcal{C}'}^k \to A_{\mathcal{C}}^k \to A_{\mathcal{C}''}^{k-1} \to 0, \tag{5.2}$$

where the second map is the residue at $\tilde{H}_{j_0}$.

Proof. The lemma follows from the fact that $A_{\mathcal{C}}^k, A_{\mathcal{C}'}^k, A_{\mathcal{C}''}^{k-1}$ are isomorphic to the top degree components of the Orlik-Solomon algebras of central arrangements. $\square$

Lemma 5.6.

(i) *For $\ell \neq k$ we have $H^\ell(\tilde{M}_{\mathcal{C}'}; \mathcal{L}_{\boldsymbol{w}}) = H^\ell(\tilde{M}_{\mathcal{C}}; \mathcal{L}_{\boldsymbol{w}}) = H^{\ell-1}(\tilde{M}_{\mathcal{C}''}; \mathcal{L}_{\boldsymbol{w}}) = 0$.*

(ii) *Consider the following diagram*

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & A_{\mathcal{C}'}^k & \longrightarrow & A_{\mathcal{C}}^k & \longrightarrow & A_{\mathcal{C}''}^{k-1} & \longrightarrow & 0 \\
& & \downarrow {\scriptstyle \iota_{\mathcal{C}'}} & & \downarrow {\scriptstyle \iota_{\mathcal{C}}} & & \downarrow {\scriptstyle \iota_{\mathcal{C}''}} & & \\
0 \longrightarrow & H^k(\tilde{M}_{\mathcal{C}'}; \mathcal{L}_{\boldsymbol{w}}) & \longrightarrow & H^k(\tilde{M}_{\mathcal{C}}; \mathcal{L}_{\boldsymbol{w}}) & \longrightarrow & H^{k-1}(\tilde{M}_{\mathcal{C}''}; \mathcal{L}_{\boldsymbol{w}}) & \longrightarrow 0
\end{array}
$$

where the top horizontal sequence is the sequence (5.2), *the homomorphisms of the bottom horizontal sequence are the homomorphisms of the exact sequence of the pair* $\tilde{M}_{\mathcal{C}} \subset \tilde{M}_{\mathcal{C}'}$, *cf. Lemma* 4.1. *Then the diagram is commutative, the horizontal sequence is exact and the vertical homomorphisms are isomorphisms.*

Proof. The proof of this lemma is similar to the corresponding proofs in Section 5.4 of [6]. Namely, using Lemma 4.1, one proves that there is a cohomology long exact sequence

$$\cdots \to H^\ell(\tilde{M}_{\mathcal{C}'}; \mathcal{L}_{\boldsymbol{w}}) \to H^\ell(\tilde{M}_{\mathcal{C}}; \mathcal{L}_{\boldsymbol{w}})$$
$$\to H^{\ell-1}(\tilde{M}_{\mathcal{C}''}; \mathcal{L}_{\boldsymbol{w}}) \to H^{\ell+1}(\tilde{M}_{\mathcal{C}'}; \mathcal{L}_{\boldsymbol{w}}) \to \ldots,$$

cf. Corollary 5.81 in [6]. Using the induction on k, one proves that $H^\ell(\tilde{M}_{\mathcal{C}'}; \mathcal{L}_{\boldsymbol{w}}) \simeq H^\ell(\tilde{M}_{\mathcal{C}}; \mathcal{L}_{\boldsymbol{w}})$ if $\ell \neq k$. Using the induction on the number of hyperplanes in $\mathcal{C}$, one concludes that $H^\ell(\tilde{M}_{\mathcal{C}}; \mathcal{L}_{\boldsymbol{w}}) = 0$ if $\ell \neq k$ and one gets an exact sequence $0 \to H^k(\tilde{M}_{\mathcal{C}'}; \mathcal{L}_{\boldsymbol{w}}) \to H^k(\tilde{M}_{\mathcal{C}}; \mathcal{L}_{\boldsymbol{w}}) \to H^{k-1}(\tilde{M}_{\mathcal{C}''}; \mathcal{L}_{\boldsymbol{w}}) \to 0$. Then using the double induction on k and the number of hyperplanes in $\mathcal{C}$ one gets the second statement of Lemma 5.6.

Lemma 5.6 implies Theorem 5.2.

Theorem 5.2 implies Theorems 3.3 and 2.1. $\qquad\qquad\square$

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